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The Extended Graetz Problem with Prescribed Wall Flux

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An analytical solution is obtained to the extended Graetz problem with prescribed wall flux, based on a selfadjoint formalism resulting from a decomposition of the convective diffusion equation into a pair of first-order partial differential equations. The solution obtained is simple, computationally efficient and in striking contrast with incomplete numerical efforts in the past.

SCOPE

The joint effects of convective and molecular transport play an important role in all contacting operations, the purpose of which is the selective movement of mass and/or energy from one phase to another. The analysis of such transport depends on the solution of the so-called convective, diffusion, differential equations subject to suitable boundary conditions. Specifically, the transport of mass or energy in fully developed laminar flow through a circular tube (or between parallel plates) has been of traditional interest. This problem, without accounting for axial molecular transport, has been called the Graetz problem. The extension of the Graetz problem to include axial conduction or diffusion, which becomes necessary for small values of an axial Peclet number, is important, for example, in dealing with heat transfer in liquid metals. Papoutsakis, Ramkrishna and Lim (1980) have reviewed past efforts in dealing with the foregoing extended Graetz problem and have provided an analytical solution to the case where the tube-wall temperature is specified. This solution, which is in sharp contrast with the approximate methods employed previously, depends on the use of a selfadjoint formalism employed by Ramkrishna and Amundson (1979) to solve certain apparently nonselfadjoint problems in heat conduction.

The present paper is an extension of the work of Papoutsakis et al. (1980) to include the case of the Neumann boundary condition for the extended Graetz problem, that is, where the energy flux at the tube wall has been specified.

CONCLUSIONS AND SIGNIFICANCE

The present work produces an analytical solution to the extended Graetz problem with finite and infinite energy or mass exchange sections and prescribed wall energy or mass fluxes, with an arbitrary number of discontinuities. The solution obtained is as simple in form and efficient computationally as the solution of the corresponding classical Graetz problem. Extensions of the solution methodology to problems with more general boundary conditions can be made in a straightforward manner.

The effect of the finite heating section length, although most important in the low Peclet number range, is significant even at Peclet numbers higher than 30, particularly close to the wall

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and /or to the end of the heating section. Although the Peclet number is the sole deciding factor for whether or not axial conduction or diffusion deserves inclusion, no global Peclet number criteria may be set for all problems. Indeed, particular interests for the solution and different wall flux regimes largely influence these criteria.

The assumption usually made in the analysis of entrance region heat or mass transfer problems, is that of negligible axial thermal conduction or diffusion. Under those assumptions and with Dirichlet wall boundary conditions (uniform wall temperature), the heat transfer problem is known as the classical Graetz problem. The extended Graetz problem refers to the original Graetz problem including axial conduction and with any wall boundary conditions. Indeed, axial heat conduction cannot be always ignored, particularly for low Prandtl number fluids, such as liquid metals, or small Reynolds number flows such as with viscous fluids. There also exist low Peclet number mass transfer problems (Tan and Hsu, 1972; Davis et al., 1974). An analytical solution of the extended Graetz problem would indeed be of both practical and theoretical importance.

Recently, Papoutsakis, Ramkrishna and Lim (1980) have discussed past efforts in dealing with the extended Graetz problem which have been largely approximate. Papoutsakis et al. (1980) obtained an exact analytical solution to the problem with prescribed wall temperature (Dirichlet problem). The objective of the present paper is to provide the solution to the extended Graetz problem for the case in which the energy or mass flux at the wall is specified. The solution obtained by Papoutsakis et al. (1980) for the Dirichlet problem was based in a selfadjoint formalism obtained by decomposing the energy equation into a pair of first-order partial differential equations (see, for example, Ramkrishna and Admundson, (1979). The decomposition can be accomplished by defining an axial energy flow function as done by Papoutsakis et al. (1980).

It is important to recognize that the extended Graetz problem for a prescribed wall flux (the Neumann problem) has some peculiarities that distinguish it from the Dirichlet problem. It is not a priori clear that an infinite heat exchange section can lead to a solution of the problem, because the role of axial conduction is to promote energy transfer from the progressively heating downstream section into the upstream region. Indeed, a constant heat flux condition at the wall over the semi-infinite section is itself suspect in this regard; it will turn out, however, tht the temperture change in the fluid becomes eventuaally linear, making the axial conduction term negligible. Thus, the elliptic equation becomes asymptotically parabolic.

Since from a practical viewpoint only a finite length of the energy transfer section is allowed, its role in determining the temperature profile becomes an important factor in the analysis. The mathematical implication of the foregoing is that the tube, considered as infinite in extent, features two discontinuities in the energy flux at the boundary. Thus, the boundary energy flux is uniformly zero (that is, the tube is insulated) in the regions before and after the energy transfer section (see Figure 1). The discontinuities occur at the beginning and the end of the energy transfer section.

Here we analyze the problem with uniform heat flux and with finite or semi-infinite heating sections.

ANALYSIS

Formalism

We consider fully developed laminar flow of a fluid in a tube with some specified velocity profile $v_z(r)$, entering with a uniform temperature profile T_0 at $z=-\infty$. The tube is thermally insulated for $z<0$ and is uniformly heated over the whole or a part (of length z_1) of the tube downstream (see Figure 1). The energy equation and the boundary conditions, assuming constant physical properties, negligible viscous dissipation and axisymmetry, are given by

$$-\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - k\frac{\partial^2 T}{\partial z^2} + \rho c_p\, v_z(r)\frac{\partial T}{\partial z} = 0 \quad \underset{-\infty<z<\infty}{0<r<R} \tag{1}$$

$$T(-\infty, r) = T_0 \tag{2}$$

$$\frac{\partial T}{\partial r}(z,0) = 0 \tag{3}$$

$$k\frac{\partial T}{\partial r}(z,R) = \begin{cases} q_w & 0\le z<z_1 \quad (4a) \\ 0 & z<0,\quad z>z_1 \quad (4b) \end{cases}$$

If the entire downstream section of the tube is heated, $z_1=\infty$. In dimensionless form, the mathematical problem is given by

$$-\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\theta}{\partial\eta}\right) - \frac{1}{Pe^2}\frac{\partial^2\theta}{\partial\zeta^2} + v(\eta)\frac{\partial\theta}{\partial\zeta} = 0 \quad \underset{-\infty<\zeta<\infty}{0<\eta<1} \tag{5}$$

$$\theta(-\infty,\eta) = 0 \tag{6}$$

$$\frac{\partial\theta}{\partial\eta}(\zeta,0) = 0 \tag{7}$$

$$\frac{\partial\theta}{\partial\eta}(\zeta,1) = \begin{cases} 1 & 0\le\zeta<Z_1 \quad (8a) \\ 0 & 0<\zeta,\quad \zeta>Z_1 \quad (8b) \end{cases}$$

where

$$\theta \equiv \frac{T-T_0}{q_w R/k},\quad \zeta \equiv \frac{z}{PeR},\quad \eta \equiv \frac{r}{R},\quad Pe \equiv \frac{\rho c_p V R}{k}$$

$$v(\eta) \equiv \frac{v_z(r)}{V},\quad Z_1 \equiv \frac{Z_1}{PeR},\quad V \equiv \text{some characteristic velocity}$$

It is possible to decompose Equation (1) into a pair of first-order partial differential equations, following a method used by Ramkrishna and Amundson (1979) for solving problems in heat conduction. The decomposition procedure can also be performed from purely physical considerations.

We define a function $S(z,r)$, which may be called the axial energy flow through a cross-sectional area of radius r concentric with the tube cross section Mathematically

$$S(z,r) = \int_0^r\left[-k\frac{\partial T}{\partial z} + \rho v_z c_p T\right]2\pi r'dr' \tag{9}$$

In Equation (9), the quantity in square brackets in the integrand is the axial energy flux comprising the conductive and convective contributions. The convective-diffusion process can now be looked upon as a pair of differential equations describing how the axial energy flow varies with z and r. Thus we have

$$\frac{\partial S}{\partial z} = -2\pi r\left[-k\frac{\partial T}{\partial r}\right] \tag{10}$$

which reflects the fact that the axial energy flow changes with axial distance because of radial diffusion. Differentiating (9), we have

$$\frac{\partial S}{\partial r} = \left[-k\frac{\partial T}{\partial z} + \rho v_z c_p T\right]2\pi r \tag{11}$$

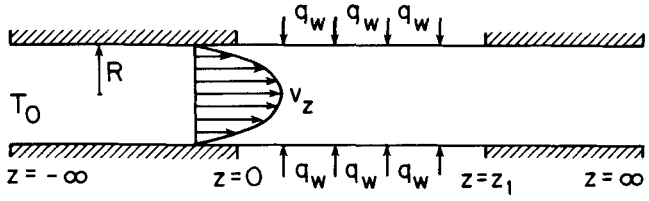

Figure 1. Sketch of the physical situation described by the extended Graetz problem with a heating section of finite length z_1.

It is readily seen that if $S(z,r)$ is eliminated between Equations (10) and (11), the result is the convective-diffusion Equation (1). Equation (9) also yields

$$S(z,0) = 0 \tag{12}$$

and

$$\lim_{z\to-\infty} S(z,r) = \int_0^r \rho c_p v_z T_0 \, 2\pi r' dr' \tag{13}$$

From an energy balance for the whole cross section of the tube between z and $z+dz$, we obtain

$$S(z+dz,R) - S(z,R) = 2\pi R q_W dz, \quad 0\le z<z_1 \tag{14}$$

and, hence

$$\frac{dS}{dz}(z,R) = 2\pi R q_W, \quad 0\le z<z_1 \tag{15}$$

and also

$$\frac{dS}{dz}(z,R) = 0 \quad , \quad z<0, \quad z>z_1 \tag{16}$$

Since

$$S(0-,R) = S(-\infty,R) \tag{17}$$

from (15) and (16) we obtain

$$S(z,R) = S(-\infty,R) \qquad z\le 0 \tag{18a}$$

$$S(z,R) = 2\pi R q_W z + S(-\infty,R) \qquad 0\le z\le z_1 \tag{18b}$$

$$S(z,R) = 2\pi R q_W z_1 + S(-\infty,R) \qquad z\ge z_1 \tag{18c}$$

To switch to dimensionless variables, we define

$$\Sigma(\zeta,\eta) \equiv \frac{k}{\pi R^2 \rho c_p V q_W R}\left\{S(z,r) - \int_0^r \rho c_p v_z T_0 2\pi r' dr'\right\} \tag{19}$$

Then, we may write from (9)

$$\Sigma(\zeta,\eta) = \int_0^\eta \left[v(\eta')\theta - \frac{1}{Pe^2}\frac{\partial\theta}{\partial\zeta}\right] 2\eta' d\eta' \tag{20}$$

The dimensionless versions of Equations (10) and (11), with a slight rearrangement of the former, may be written respectively, as

$$\frac{\partial\theta}{\partial\zeta} = Pe^2 v(\eta)\theta - \frac{Pe^2}{2\eta}\frac{\partial\Sigma}{\partial\eta} \tag{21}$$

$$\frac{\partial\Sigma}{\partial\zeta} = 2\eta\frac{\partial\theta}{\partial\eta} \tag{22}$$

Further, from Equations (12), (13) and (18), we readily conclude that

$$\Sigma(\zeta,0) = 0 \qquad -\infty<\zeta<\infty \tag{23}$$

$$\lim_{\zeta\to-\infty} \Sigma(\zeta,\eta) = 0 \qquad 0<\eta<1 \tag{24}$$

$$\Sigma(\zeta,1) = \begin{cases} 0 & -\infty<\zeta\le 0 \\ 2\zeta & 0\le\zeta\le Z_1 \\ 2Z_1 & Z_1\le\zeta<\infty \end{cases} \tag{25}$$

We have now recast the problem as given by the pair of partial differential Equations (21) and (22) in the two functions $\theta(\zeta,\eta)$ and $\Sigma(\zeta,\eta)$ with boundary conditions given by (6), (23), (24) and (25). Note that (7) and (8) were not included in the boundary conditions since they are implied, respectively, by Equations (23) and (25). Equations (21) and (22) may now be written in matrix notation as

$$\frac{\partial}{\partial\zeta}\begin{bmatrix}\theta(\zeta,\eta)\\ \Sigma(\zeta,\eta)\end{bmatrix} = \begin{bmatrix} Pe^2 v(\eta) & -\dfrac{Pe^2}{2\eta}\dfrac{\partial}{\partial\eta} \\ 2\eta\dfrac{\partial}{\partial\eta} & 0 \end{bmatrix}\begin{bmatrix}\theta(\zeta,\eta)\\ \Sigma(\zeta,\eta)\end{bmatrix} \tag{26}$$

To achieve some notational brevity, we denote by $\underset{\sim}{L}$ the differential operator denoted in matrix form in Equation (26) and by $\underset{\sim}{F}(\zeta)$ the two-component vector appearing on both sides of Equation (26). Thus, for each ζ, this vector may be looked upon as an element of the linear space $\mathscr{H}_1 \oplus \mathscr{H}_2$ which consists of ordered pairs of elements from the respective spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, both of which represent the space of functions of η square integrable in $[0,1]$. These spaces are clearly defined in appendix A. Equation (26) is now written as

$$\frac{d}{d\zeta}\underset{\sim}{F}(\zeta) = \underset{\sim}{L}\underset{\sim}{F}(\zeta) \tag{27}$$

The solution of the original problem is now reduced to that of solving (27) subject to boundary conditions (6), (23), (24) and (25).

The reason however for defining the operator $\underset{\sim}{L}$ is that it gives rise to a selfadjoint problem, even if the original convective-diffusion operator is nonselfadjoint. Indeed, if we define the inner product between two vectors

$$\underset{\sim}{\phi} = \begin{bmatrix}\phi_1(\eta)\\ \phi_2(\eta)\end{bmatrix}, \quad \underset{\sim}{\psi} = \begin{bmatrix}\psi_1(\eta)\\ \psi_2(\eta)\end{bmatrix} \quad \text{as}$$

$$<\underset{\sim}{\phi},\underset{\sim}{\psi}> = \int_0^1 \left[\frac{4}{Pe^2}\phi_1(\eta)\psi_1(\eta)\eta + \phi_2(\eta)\psi_2(\eta)\frac{1}{\eta}\right] d\eta \tag{28}$$

and the following domain for $\underset{\sim}{L}$

$$D(\underset{\sim}{L}) = \{\underset{\sim}{\phi}\epsilon\mathscr{H}: \underset{\sim}{L}\underset{\sim}{\phi}(\text{exists and})\epsilon\mathscr{H},\ \phi_2(0)=\phi_2(1)=0\} \tag{29}$$

it is shown in appendix A that $\underset{\sim}{L}$ is a symmetric operator in the Hilbert space $\mathscr{H}$ of interest. Thus, the selfadjoint (or Sturm-Liouville) eigenvalue problem is given by

$$\underset{\sim}{L}\underset{\sim}{\phi}_j = \lambda_j \underset{\sim}{\phi}_j \tag{30}$$

from which we may obtain

$$Pe^2 v(\eta)\phi_{j1}(\eta) - \frac{Pe^2}{2\eta}\phi'_{j2}(\eta) = \lambda_j \phi_{j1}(\eta) \tag{31}$$

$$2\eta\,\phi'_{j1}(\eta) = \lambda_j \phi_{j2}(\eta) \tag{32}$$

Therefore, the eigenvalue problem (30) implies, through (31), (32) and (29), that

$$\frac{1}{\eta}(\eta\phi'_{j1})' + \left[\frac{\lambda_j}{Pe^2} - v(\eta)\right]\lambda_j \phi_{j1} = 0 \tag{33}$$

subject to the boundary conditions $\phi'_{j1}(0) = \phi'_{j1}(1) = 0$. $\underset{\sim}{L}$ is neither a positive definite operator nor a negative definite operator and in fact possesses both positive eigenvalues $\{\lambda^+_j\}$ with corresponding eigenvectors $\{\underset{\sim}{\phi}^+_j\}$ and negative eigenvalues $\{\lambda^-_j\}$ with eigenvectors $\{\underset{\sim}{\phi}^-_j\}$. The two sets of eigenvectors, normalized according to (28), together constitute an orthonormal basis in $\mathscr{H}$. It is worth noting that the eigenvalue problem (33) is the same as the eigenvalue problem obtained in the past (Hsu, 1967; Pirkle and Sigillito, 1972; Tan and Hsu, 1972; Michelsen and Villadsen, 1974). The analysis herein justifies rigorously the implicit assumption made by the foregoing authors that the λ'_js are real since they are, in fact, the eigenvalues of a selfadjoint problem. With the exception of Michelsen and Villadsen (1974), the existence of both positive and negative eigenvalues was not commonly recognized.

Before we proceed to the solution of the problem, the following expansion theorem is obtained from the eigenvalue problem, for any vector $\underset{\sim}{f}\epsilon\mathscr{H}$:

$$\underset{\sim}{f} = \sum_{j=1}^{\infty}\left[\frac{<\underset{\sim}{f},\underset{\sim}{\phi}^+_j>}{\|\underset{\sim}{\phi}^+_j\|^2}\underset{\sim}{\phi}^+_j + \frac{<\underset{\sim}{f},\underset{\sim}{\phi}^-_j>}{\|\underset{\sim}{\phi}^-_j\|^2}\underset{\sim}{\phi}^-_j\right] \equiv \sum_{j=1}^{\infty}\frac{<\underset{\sim}{f},\underset{\sim}{\phi}_j>}{\|\underset{\sim}{\phi}_j\|^2}\underset{\sim}{\phi}_j \tag{34}$$

Here

$$\|\underset{\sim}{\phi}\|^2 \equiv <\underset{\sim}{\phi},\underset{\sim}{\phi}> \tag{35}$$

The Solution

The solution of the problem, $\underset{\sim}{F}(\zeta)$, is obtained in the form of series (34). To that effect, the inner products in the expansion

coefficients of (34) must be determined.

From (30), using (29), and as in the proof of appendix A, we may obtain

$$<\underset{\sim}{L}\underset{\sim}{F},\underset{\sim}{\phi}_j> = \lambda_j <\underset{\sim}{F},\underset{\sim}{\phi}_j> - 2\phi_{j1}(1)\,\Sigma(\zeta,1) \tag{36}$$

Now, taking the inner product of both sides of (27) with $\underset{\sim}{\phi}_j$ and using (36), we have

$$\frac{d}{d\zeta}<\underset{\sim}{F},\underset{\sim}{\phi}_j> = \lambda_j<\underset{\sim}{F},\underset{\sim}{\phi}_j> - 2\phi_{j1}(1)\,\Sigma(\zeta,1) \tag{37}$$

Note that (37) is a differential equation with the scalar inner product $<\underset{\sim}{F},\underset{\sim}{\phi}_j>$ as the unknown and also that $\Sigma(\zeta,1)$ is given by (25). Thus, solving (37) separately for λ^+_j and λ^-_j, we may obtain

$$<\underset{\sim}{F},\underset{\sim}{\phi}^+_j> = C^+_j e^{\lambda^+_j\zeta} + 2\phi^+_{j1}(1)\int_\zeta^\infty \Sigma(\zeta',1)e^{\lambda^+_j(\zeta-\zeta')}d\zeta' \tag{38}$$

$$<\underset{\sim}{F},\underset{\sim}{\phi}^-_j> = C^-_j e^{\lambda^-_j\zeta} - 2\phi^-_{j1}(1)\int_{-\infty}^\zeta \Sigma(\zeta',1)e^{\zeta^-_j(\zeta-\zeta')}d\zeta' \tag{39}$$

In appendix B it is shown that $\lambda_0 = 0$ is not an eigenvalue. Since the solution must be finite at both $+\infty$ and $-\infty$, we have

$$C^-_j = C^+_j = 0 \tag{40}$$

The integrations indicated in (38) and (39) may be carried out easily, although lengthily, using Equation (25). Using, afterwards, the first vector components of Equation (34) with $\underset{\sim}{f} = \underset{\sim}{F}(\zeta)$, we may obtain the solution in the form

$$\theta(\zeta,\eta) = 4\sum_{j=1}^\infty \frac{\phi^+_{j1}(1)\,e^{\lambda^+_j\zeta}(1-e^{-\lambda^+_j Z_1})}{\lambda^{+2}_j\|\underset{\sim}{\phi}^+_j\|^2}\,\phi^+_{j1}(\eta) \quad \zeta<0 \tag{41}$$

$$\theta(\zeta,\eta) = 4\sum_{j=1}^\infty \frac{\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda^2_j\|\underset{\sim}{\phi}_j\|^2} + 4\zeta\sum_{j=1}^\infty \frac{\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda_j\|\underset{\sim}{\phi}_j\|^2} - 4\sum_{j=1}^\infty \frac{\phi^+_{j1}(1)\phi^+_{j1}(\eta)e^{\lambda^+_j(\zeta-Z_1)}}{\lambda^{+2}_j\|\underset{\sim}{\phi}^+_j\|^2} - 4\sum_{j=1}^\infty \frac{\phi^-_{j1}(1)\phi^-_{j1}(\eta)e^{\lambda^-_j\zeta}}{\lambda^{-2}_j\|\underset{\sim}{\phi}^-_j\|^2} \quad 0<\zeta<Z_1 \tag{42}$$

$$\theta(\zeta,\eta) = 4Z_1\sum_{j=1}^\infty \frac{\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda_j\|\underset{\sim}{\phi}_j\|^2} - 4\sum_{j=1}^\infty \frac{\phi^-_{j1}(1)\phi^-_{j1}(\eta)e^{\lambda^-_j\zeta}(1-e^{-\lambda^-_j Z_1})}{\lambda^{-2}_j\|\underset{\sim}{\phi}^-_j\|^2} \quad \zeta>Z_1 \tag{43}$$

It is shown in appendix C that

$$4\sum_{j=1}^\infty \frac{\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda_j\|\underset{\sim}{\phi}_j\|^2} = 4 \tag{44}$$

and

$$4\sum_{j=1}^\infty \frac{\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda^2_j\|\underset{\sim}{\phi}_j\|^2} = \eta^2 - \frac{\eta^4}{4} + \frac{8}{Pe^2} - \frac{7}{24} \tag{45}$$

In addition, it can be shown (Papoutsakis, 1979; Papoutsakis et al., 1980) that

$$\|\underset{\sim}{\phi}_j\|^2 = -\frac{2}{\mu_j^2}\phi_{j1}(1)\frac{d}{d\mu_j}\left(\frac{\phi'_{j1}(1)}{\mu_j}\right) \tag{46}$$

where

$$\mu_j = |\lambda_j|^{1/2} \tag{47}$$

If we now denote by A^+_j, A^-_j the expansion coefficients in (41) to (43) as follows, we obtain, using (46)

$$A^+_j = \frac{4\phi^+_{j1}(1)}{\lambda^{+2}_j\|\underset{\sim}{\phi}^+_j\|^2} = -2\left[\mu^{+2}_j\frac{d}{d\mu^+_j}\left(\frac{\phi'^+_{j1}(1)}{\mu^+_j}\right)\right]^{-1} \tag{48}$$

$$A^-_j = -\frac{4\phi^-_{j1}(1)}{\lambda^{-2}_j\|\underset{\sim}{\phi}^-_j\|^2} = 2\left[\mu^{-2}_j\frac{d}{d\mu^-_j}\left(\frac{\phi'^-_{j1}(1)}{\mu^-_j}\right)\right]^{-1} \tag{49}$$

By virtue of (44) to (48), we may now write the solution (41) to (43) in the final form:

$$\theta(\zeta,\eta) = \sum_{j=1}^\infty A^+_j\,e^{\lambda^+_j\zeta}(1-e^{-\lambda^+_j Z_1})\phi^+_{j1}(\eta) \quad \zeta<0 \tag{50}$$

$$\theta(\zeta,\eta) = 4\zeta + \eta^2 - \frac{\eta^4}{4} + \frac{8}{Pe^2} - \frac{7}{24} - \sum_{j=1}^\infty A^+_j e^{\lambda^+_j(\zeta-Z_1)}\phi^+_{j1}(\eta) + \sum_{j=1}^\infty A^-_j e^{\lambda^-_j\zeta}\phi^-_{j1}(\eta) \quad 0<\zeta<Z_1 \tag{51}$$

$$\theta(\zeta,\eta) = 4Z_1 + \sum_{j=1}^\infty A^-_j\,e^{\lambda^-_j\zeta}(1-e^{-\lambda^-_j Z_1})\,\phi^-_{j1}(\eta) \quad \zeta>Z_1 \tag{52}$$

Note from (47) that $\mu^-_j>0$. The continuity of $\theta(\zeta,\eta)$ at $\zeta = 0$ and $\zeta = Z_1$ may be most easily checked from Equations (41) to (43). Boundary conditions (6), (23) to (25) [and hence (7) and (8)] are obviously satisfied by the solution (50) to (52).

For $Z_1 = \infty$, we may specifically obtain the solution

$$\theta(\zeta,\eta) = \sum_{j=1}^\infty A^+_j\,e^{\lambda^+_j\zeta}\phi^+_{j1}(\eta) \quad \zeta<0 \tag{53}$$

$$\theta(\zeta,\eta) = 4\zeta + \eta^2 - \frac{\eta^4}{4} + \frac{8}{Pe^2} - \frac{7}{24} + \sum_{j=1}^\infty A^-_j\,e^{\lambda^-_j\zeta}\phi^-_{j1}(\eta) \quad \zeta>0 \tag{54}$$

The solution for $\zeta<0$ in either case, Equation (50) or (53), does not involve the eigenvectors ϕ^-_j, since they are orthogonal to the solution of the problem $\underset{\sim}{F}(\zeta)$ when $\zeta < 0$. Indeed, from Equations (39), (40) and (25), it is immediately obtained that

$$<\underset{\sim}{F},\underset{\sim}{\phi}^-_j> = 0 \qquad \zeta<0\,,\ \text{all } j \tag{55}$$

However, the solution for $\zeta > 0$, Equations (51), (52) or (54), utilizes the eigenvectors $\underset{\sim}{\phi}^+_j$, although implicitly for (52) and (54), as is evident from Equations (44) and (45); the eigenvectors $\underset{\sim}{\phi}^+_j$ are not orthogonal to $\underset{\sim}{F}(\zeta)$ for $\zeta > 0$.

Note that the solution of either problem, Equations (50) to (52) or (53) and (54), does not depend explicitly upon the Peclet number, other than the term $8/Pe^2$, and although operator $\underset{\sim}{L}$ is not defined for $Pe = \infty$, we may take as high a Pe as desired. Thus, Equations (51) to (52) or (53) and (54) naturally represent the solution of the parabolic problem (where $Pe = \infty$) if the corresponding eigenvalues are determined from (33) with $Pe = \infty$. Notice that for $Pe \to \infty$, all λ^+_j's go to infinity, and from Equations (50) and (48)

$$\theta(\zeta,\eta) = 0 \quad \zeta < 0 \tag{56}$$

and also that Equation (51) reduces to Equation (54), as expected. Indeed, in the parabolic problem, the length of the heating section does not affect the temperature field in that section, since there is no axial heat conduction term in the energy equation. A mathematically interesting solution of the parabolic problem and some interesting comparisons between the solutions of the parabolic and the elliptic problems may be found in the thesis by Papoutsakis (1979).

Two quantities of practical importance, the bulk or average temperature T_b and the Nusselt number, are usually defined as

$$T_b = \int_0^R T\,v_z(r)rdr \Big/ \int_0^R v_z(r)rdr \tag{57}$$

$$Nu = -\frac{k\left(\frac{\partial T}{\partial r}\right)_{r=R}}{T_b-T_w}\cdot\frac{2R}{k} \tag{58}$$

which, in dimensionless form, may be written as

$$\theta_b = 4\int_0^1 \theta v(\eta)\eta d\eta \tag{59}$$

$$Nu = -\frac{2\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1}}{\theta_b-\theta_w} \tag{60}$$

Using Equations (50) to (52) and (33), we may eventually obtain from (59) and (60)

$$\theta_b = \sum_{j=1}^{\infty}\frac{4A^+_j\lambda^+_j}{Pe^2}e^{\lambda^+_j\zeta}(1-e^{-\lambda^+_jZ_1})\int_0^1\phi^+_{j1}\eta d\eta, \quad \zeta<0 \tag{61}$$

$$\theta_b = 4\zeta + \frac{8}{Pe^2} + \sum_{j=1}^{\infty}\frac{4A^-_j\lambda^-_j e^{\lambda^-_j\zeta}}{Pe^2}\int_0^1\phi^-_{j1}\eta d\eta - \sum_{j=1}^{\infty}\frac{4A^+_j\lambda^+_j e^{\lambda^+_j(\zeta-Z_1)}}{Pe^2}\int_0^1\phi^+_{j1}\eta d\eta, \quad 0<\zeta<Z_1 \tag{62}$$

$$\theta_b = 4Z_1 + \sum_{j=1}^{\infty}\frac{4A^-_j\lambda^-_j e^{\lambda^-_j\zeta}(1-e^{-\lambda^-_jZ_1})}{Pe^2}\int_0^1\phi^-_{j1}\eta d\eta, \quad \zeta>Z_1 \tag{63}$$

$$Nu = 2\left[\sum_{j=1}^{\infty}A^+_j e^{\lambda^+_j\zeta}(1-e^{-\lambda^+_jZ_1})\left\{\phi^+_{j1}(1) - \frac{4\lambda^+_j}{Pe^2}\int_0^1\phi^+_{j1}\eta d\eta\right\}\right]^{-1} \quad \zeta<0 \tag{64}$$

$$Nu = 2\left[\frac{11}{24} + \sum_{j=1}^{\infty}A^-_j e^{\lambda^-_j\zeta}\left\{\phi^-_{j1}(1) - \frac{4\lambda^-_j}{Pe^2}\int_0^1\phi^-_{j1}\eta d\eta\right\} - \sum_{j=1}^{\infty}A^+_j e^{\lambda^+_j(\zeta-Z_1)}\cdot\left\{\phi^+_{j1}(1) - \frac{4\lambda^+_j}{Pe^2}\int_0^1\phi^+_{j1}\eta d\eta\right\}\right]^{-1} \quad 0<\zeta<Z_1 \tag{65}$$

$$Nu = 2\left[\sum_{j=1}^{\infty}A^-_j e^{\lambda^-_j\zeta}(1-e^{-\lambda^-_jZ_1})\left\{\phi^-_j(1) - \frac{4\lambda^-_j}{Pe^2}\int_0^1\phi^-_{j1}\eta d\eta\right\}\right]^{-1} \quad \zeta>Z_1 \tag{66}$$

For the case $Z_1 = \infty$, the corresponding quantities are obtained from (61) to (66) by taking the limits for $Z_1 \to \infty$. Note from (65) that for $Z_1 \to \infty$, the asymptotic Nusselt number is $2 \cdot 24/11 = 4.3636$ for any Peclet number.

COMPUTATIONAL ASPECTS

For the parabolic velocity profile, the eigenvectors ϕ_j may be computed using Kummer's functions, after transforming Equation (33) into Kummer's equation (Slater, 1960, chapter 5). Thus, we have

$$\phi^-_{j1}(\eta) = \exp\left(-\frac{\mu^-_j}{2}\eta^2\right)M(a^-_j, 1, \mu^-_j\eta^2) \tag{67}$$

$$\phi^+_{j1}(\eta) = \exp\left(-\frac{i\mu^+_j}{2}\eta^2\right)M(a^+_j, 1, i\mu^+_j\eta^2) \tag{68}$$

where $M(\ ,\ ,\ )$ is Kummer's function (Slater, 1960), and

$$a^-_j = \frac{1}{2}\left(1-\frac{\mu^-_j}{2}-\frac{\mu^{-3}_j}{2Pe^2}\right) \tag{69}$$

$$a^+_j = \frac{1}{2}\left(1-\frac{i\mu^+_j}{2}-\frac{(i\mu^+_j)^3}{2Pe^2}\right) \tag{70}$$

while the eigenvalues are calculated from the characteristic equation

$$2M'(a_j, 1, \mu_j) - M(a_j, 1, \mu_j) = 0 \tag{71}$$

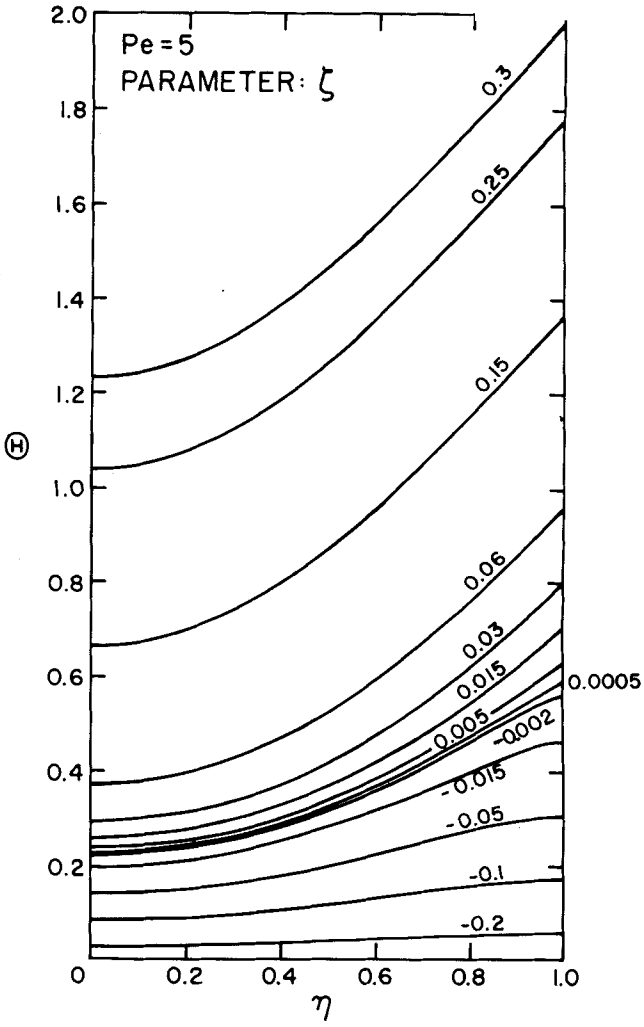

Figure 2. Radial profiles of dimensionless temperature for various axial distances; $Pe = 5$, $Z_1 = \infty$.

obtained from B.C. $\phi'_{j1}(1) = 0$. The real positive roots of (71) provide the set $\{\mu^-_j\}$, while the positive imaginary parts of the purely imaginary roots of (71) provide the set $\{\mu^+_j\}$. The computation of the eigenquantities through Equations (67) to (71) is very efficient and accurate, allowing any number of them to be computed to any desired accuracy (Papoutsakis, 1979). All the computations were performed on the Purdue University CDC 6600 computer. Thus, for example, the computation of 20 μ^-_j's, with ten significant figures, required 1 to 2 CP s, while for the μ^+_j's, a time of 4 to 6 CP s was required owing to the complex arithmetic as is evident from Equations (68), (70) and (71). The derivatives appearing in the expansion coefficients, Equations (48) and (49), were computed by their backward finite differences with an increment of 10^{-6}. The integrals appearing in Equations (61) to (66) may be computed analytically very efficiently owing to the fast convergence of their series representation obtained from direct integration of Equations (67) and (68).

The computation of the temperature at 273 points, typically used for any of the Figures 2 to 5, required 4 to 5 CP s, although for 546 points, the required time was 5 to 6 s owing to repeated intermediate computations. The overall computational accuracy was maintained typically to better than 10^{-3}%. By far, the largest percentage of the computation time was spent for computations involving the positive eigenvalues, because of the complex arithmetic. Although it is of interest to study the effect of the axial conduction in the upstream region ($\zeta<0$) of the tube, of practical interest is mainly the solution of the problem for $\zeta>0$, that is, Equations (51) and (54). For the latter cases, the

TABLE 1. SQUARE ROOTS OF POSITIVE EIGENVALUES μ^+_j AND THE CORRESPONDING EXPANSION COEFFICIENTS A^+_j OF EQUATION (48); PE = 5 AND 20 (XE$\pm\alpha$ MEANS X · $10^{\pm\alpha}$)

	$Pe = 5$		$Pe = 20$	
j	μ^+_j	A^+_j	μ^+_j	A^+_j
1	3.34650340560E+00	2.63707943E−01	1.06491169524E+01	1.21642668E−03
2	5.51857878180E+00	−6.43378662E−02	1.66084938275E+01	−6.72071858E−04
3	6.69077372170E+00	4.47098795E−02	1.88009390029E+01	2.84545545E−03
4	7.75144665380E+00	−2.95496693E−02	2.00578786323E+01	−6.85479729E−03
5	8.69548254370E+00	2.11055654E−02	2.11409793606E+01	9.14922281E−03
6	9.55056602490E+00	−1.59868600E−02	2.23457309157E+01	−8.76919381E−03
7	1.03368983582E+01	1.26331507E−02	2.35732586850E+01	7.83285298E−03
8	1.10684202630E+01	−1.03019497E−02	2.47801055135E+01	−6.91867716E−03
9	1.17550382641E+01	8.60640290E−03	2.59535869916E+01	6.11777056E−03
10	1.24040403590E+01	−7.32859212E−03	2.70903773047E+01	−5.43599497E−03
11	1.30209556841E+01	6.33775535E−03	2.81906702634E+01	4.85933966E−03
12	1.36100897588E+01	−5.55129947E−03	2.92560460352E+01	−4.37068516E−03
13	1.417486972 E+01	4.91480804E−03	3.028858428 E+01	3.95444985E−03
14	1.471807476 E+01	−4.39115956E−03	3.129047501 E+01	−3.59759533E−03

computation times were half or one third of the times reported above. Further computational details may be found elsewhere (Papoutsakis, 1979). That the presented solution is efficient may be judged from the information provided above. That should also be evident from the solution which has the same form as the solution of the parabolic problem, and hence, computationally, it is as simple and efficient.

RESULTS

The Eigenquantities

A rather large number of negative eigenvalues has been presented in the literature (Hsu, 1967; Pirkle and Sigillito, 1972), and more accurate values are given by Papoutsakis (1979). Here we present a sample only of positive eigenvalues with their expansion coefficients A^+_j, for $Pe = 5$ and 20 (Table 1) and a sample of expansion coefficients A^-_j for Pe=5 and 10 (Table 2). Because of the fast decrease of the expansion coefficients, particularly for $\zeta \geq 0$, the temperature field may be evaluated even at ζ=0, with a reasonably small number of terms, ten to twenty of them, depending upon the desired accuracy. This advantage is characteristic of the Neumann boundary conditions (this problem) and of the Robin boundary conditions for all physically important cases but does not exist for the Dirichlet boundary conditions (Papoutsakis et al., 1980).

TABLE 2. EXPANSION COEFFICIENTS A^-_j OF EQUATION (49); PE = 5 AND 10(XE$\pm\alpha$ MEANS X · $10^{\pm\alpha}$)

j	$Pe = 5$	$Pe = 10$
1	2.59631206E−01	3.14735964E−01
2	−9.13990043E−02	−1.11687501E−01
3	4.83325819E−02	5.76808866E−02
4	−3.07292047E−02	−3.57010280E−02
5	2.16631234E−02	2.46218982E−02
6	−1.63039047E−02	−1.82183098E−02
7	1.28345858E−02	1.41528236E−02
8	−1.04397451E−02	−1.13915094E−02
9	8.70576914E−03	9.41883917E−03
10	−7.40313911E−03	−7.95341997E−03
11	6.39543334E−03	6.83047188E−03
12	−5.59704259E−03	−5.94794763E−03
13	4.95182935E−03	5.23970647E−03
14	−4.42161911E−03	−4.66123276E−03
15	3.97964689E−03	4.18159960E−03
16	−3.60665173E−03	−3.77873172E−03
17	3.28845767E−03	3.43649993E−03
18	−3.01442649E−03	−3.14287758E−03
19	2.77646411E−03	2.88873528E−03
20	−2.56827582E−03	−2.66703432E−03

Semi-Infinite Heating Section ($Z_1 = \infty$)

Radial temperature profiles for various axial distances have been computed according to Equations (53) and (54) for Pe =5 and 30 and are shown in Figures 2 and 3, respectively. Further computational results may be found elsewhere (Papoutsakis, 1979). It would be safe to say that unless one is interested in those small distances near the beginning of the heating section, axial heat conduction may be neglected for $Pe > 30$. Hennecke (1968) suggested instead the value of $Pe = 10$ using as a criterion

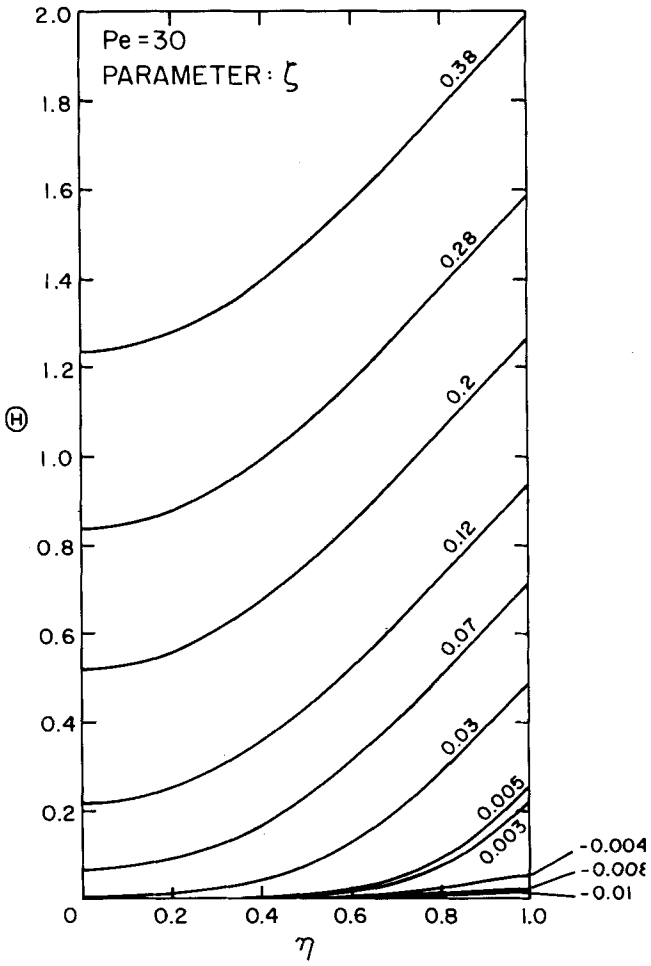

Figure 3. Radial profiles of dimensionless temperature for various axial distances; $Pe = 30$, $Z_1 = \infty$.

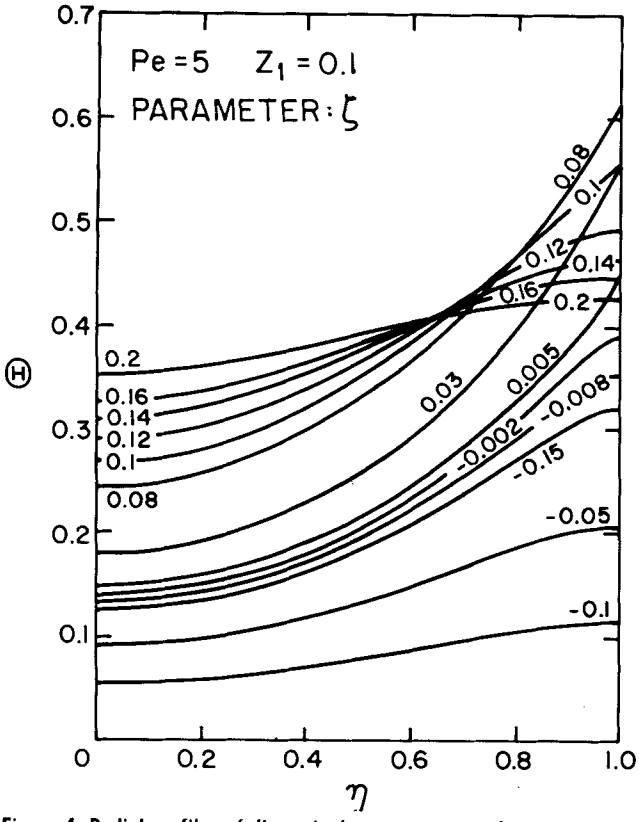

Figure 4. Radial profiles of dimensionless temperature for various axial distances; $Pe = 5$, $Z_1 = 0.1$.

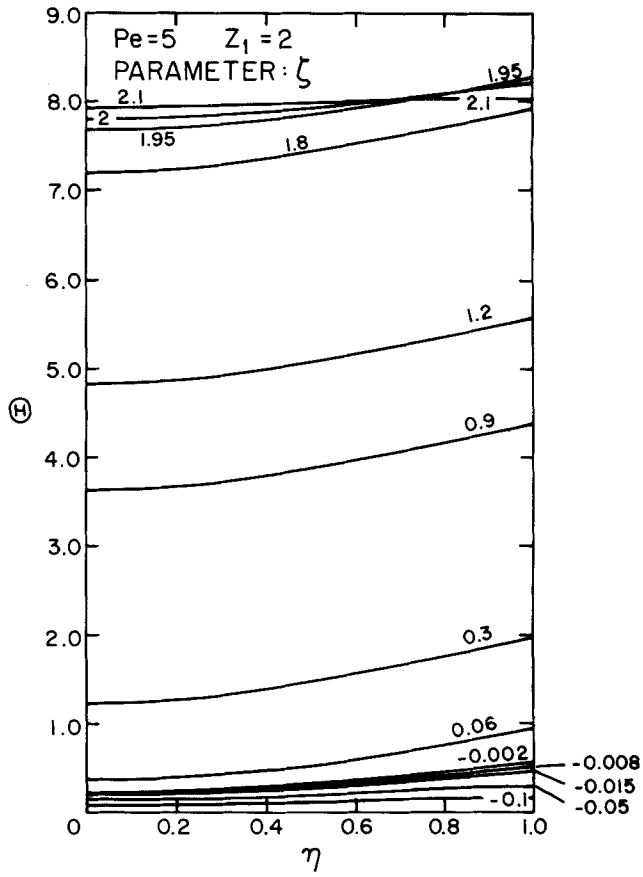

Figure 5. Radial profiles of dimensionless temperature for various axial distances; $Pe = 5$, $Z_1 = 2$.

the bulk temperature θ_b. Certainly, the Nusselt number even for $Pe = 50$ is different from the Nusselt number for $Pe = \infty$ up to one third to one fourth of a tube diameter from the beginning of the heating section. Incidentally, to the extent that the figures permit inference of the plotted values, for small Peclet numbers and axial distances, the temperature values computed with the finite difference scheme by Hennecke (1968) differ by 3 to 6% from the exact values of the solution presented here. In any case, a uniform temperature profile at $\zeta = 0$ is not established until some Peclet number higher, say, than 70.

Finite Z_1 Heating Section

The effect of the length of the heating section on the temperature field for $Pe = 5$ is shown in Figures 4 and 5 with $Z_1 = 0.1$ and 2.0, respectively. All computations have been performed according to Equations (50) to (52). The radial temperature profile at the end of the heating section ($\zeta = Z_1+$) is shown on both Figures 4 and 5. Note that now the wall flux displays two discontinuities, one at $\zeta = 0$ and one at $\zeta = Z_1$. Yet, the temperature is continuous throughout the tube. The effect of the length Z_1 on the temperature field upstream depends as expected upon the Peclet number. Thus, for $Pe = 5$ and $Z_1 = 0.1$, this effect reaches well into the negative section ($\zeta < 0$) of the tube, but for $Z_1 = 2.0$, this effect is significant only within a length of $\zeta = 0.4$ to 0.5 from the end of the heating section. It is reasonable to state that if one is not interested in the temperature field at small radial distances from the tube wall, the effect of the heating length Z_1 is most significant within ζ distances (from the end of the heating section) of 0.5, 0.3 and 0.15 for $Pe = 5$, 10 and 20, respectively. The axial length ζ_u that is required for the uniform temperature profile to be established into the downstream insulated section of the tube (such that $\zeta = Z_1 + \zeta_u$) depends not only upon the Peclet number but also upon Z_1. Thus, for $Pe = 5$ and 10, this length ζ_u ranges from 0.1 to 0.2, while for $Pe = 20$, it ranges from 0.1 to 0.15; the larger the Z_1, the smaller the ζ_u. Even at Peclet numbers higher than 30, the axial conduction effect is significant near the tube wall and towards the end of the heating section; thus, whether axial conduction deserves inclusion will depend primarily upon the length of the heating section.

DISCUSSION

The heat transfer problem in the outlet region (Hennecke, 1968) where the negative section ($\zeta < 0$) of the tube is heated with a uniform heat flux while the positive section is thermally insulated may be treated in an exactly similar manner as above. From the analysis presented herein, it is also clear that the problem with any prescribed wall flux (to suit any physically important situation), with an arbitrary number of discontinuities, may be handled in a straightforward manner. Note, in fact, that with a single set of eigenquantities and expansion coefficients, a variety of problems (for example, various heating section lengths, different flux distributions, etc.) may be analyzed with little extra computations. Such a potential for extensions and efficient simplicity are not indeed available with numerical schemes, where each problem must be treated separately from the beginning. As has been discussed above, the presented solution for the complete elliptic problem is as simple and efficient as the solution of the simplified parabolic problem ($Pe = \infty$). Hence, it is fair to suggest the use of the solution of the complete problem, for those Peclet number regimes ($30 \leq Pe \leq 100$) where there exists ambiguity regarding the importance of the axial conduction. Indeed, no global Peclet number criteria may be set for all problems regarding the inclusion of axial conduction, for particular interest and different wall flux regimes largely influence these criteria.

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APPENDIX A

Consider the Hilbert space $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ is the space of all functions $f(\eta)$ defined on [0,1] such that the Lebesque integral

$$\int_0^1 f^2(\eta)\eta d\eta < \infty$$

and $\mathcal{H}_2$ is the space of all functions $g(\eta)$ defined on [0,1] such that the Lebesque integral

$$\int_0^1 g^2(\eta)\frac{d\eta}{\eta} < \infty$$

Clearly $\mathcal{H}$ is a Hilbert space with the inner product between two vectors $\underset{\sim}{\phi}, \underset{\sim}{\psi} \in \mathcal{H}$ given by Equation (28), where $\underset{\sim}{\phi}, \underset{\sim}{\psi}$ are identified directly above Equation (28).

Let $\underset{\sim}{\phi}, \underset{\sim}{\psi} \in D(\underset{\sim}{L})$, where $D(\underset{\sim}{L})$ is given by Equation (29); then

$$<\underset{\sim}{L}\underset{\sim}{\phi}, \underset{\sim}{\psi}> - <\underset{\sim}{\phi}, \underset{\sim}{L}\underset{\sim}{\psi}> = \int_0^1 2[\phi_1\psi_2 - \phi_2\psi_1]' d\eta =$$
$$= 2[\phi_1(1)\psi_2(1) - \phi_2(1)\psi_1(1) - \phi_1(0)\psi_2(0) + \phi_2(0)\psi_1(0)] =$$
$$= 0$$

that is, $\underset{\sim}{L}$ with $D(\underset{\sim}{L})$ is a symmetric operator in $\mathcal{H}$.

APPENDIX B

We want to prove that $\lambda_0 = 0$ is not an eigenvalue of the eigenvalue problem

$$[\underset{\sim}{L}\underset{\sim}{\phi}_j = \lambda_j \underset{\sim}{\phi}_j,\ \phi_{j2}(0) = \phi_{j2}(1) = 0] \tag{B1}$$

Let $\lambda_0 = 0$ be an eigenvalue of problem (B1). From Equations (31) and (32), respectively, we obtain

$$Pe^2 v(\eta)\phi_{01}(\eta) - \frac{Pe^2}{2\eta}\phi'_{02}(\eta) = 0, \quad \text{all } \eta \tag{B2}$$

$$2\eta\,\phi'_{01}(\eta) = 0 \quad , \quad \text{all } \eta \tag{B3}$$

From (B3) we obtain

$$\phi_{01}(\eta) = \text{constant} \equiv C_1 \tag{B4}$$

If we substitute (B4) into (B2) and integrate, we get

$$\phi_{02}(\eta) = 2C_1 \int_0^\eta \eta' v(\eta') d\eta' + C_2 \tag{B5}$$

and since $\phi_{02}(0) = 0$, we have $C_2 = 0$. Hence, the integral of (B5) can by no means become zero for any possible (positive) velocity profile and $\eta = 1$. Thus, $\phi_{02}(1) = 0$ cannot be satisfied. Therefore $\lambda_0 = 0$ is *not* an eigenvalue of problem (B1). This concludes the proof.

APPENDIX C

We shall prove that

$$\sum_{j=1}^{\infty} \frac{\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda_j \|\underset{\sim}{\phi}_j\|^2} = 1 \tag{C1}$$

From the expansion theorem (34) with $\underset{\sim}{f} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we obtain using the first vector components

$$\frac{4}{Pe^2}\sum_{j=1}^{\infty} \frac{\phi_{j1}(\eta)}{\|\underset{\sim}{\phi}_j\|^2} \int_0^1 \phi_{j1}(\eta)\eta d\eta = 1 \tag{C2}$$

From (31), multiplying by η/λ_j, integrating and using (29), we have

$$\int_0^1 \phi_{j1}(\eta)\eta d\eta = \frac{Pe^2}{\lambda_j}\int_0^1 \phi_{j1}(\eta) v(\eta)\eta d\eta \tag{C3}$$

and hence (C2) may be written as

$$4\sum_{j=1}^{\infty} \frac{\phi_{j1}(\eta)}{\lambda_j \|\underset{\sim}{\phi}_j\|^2} \int_0^1 \phi_{j1}(\eta) v(\eta)\eta d\eta = 1 \tag{C4}$$

With $v(\eta) = 1 - \eta^2$, we may obtain

$$\int_0^1 \phi_{j1}(\eta) v(\eta)\eta d\eta = \int_0^1 \phi_{j1}(\eta)\eta d\eta - \int_0^1 \phi_{j1}(\eta)\eta^3 d\eta \tag{C5}$$

By virtue of (32) and integrating by parts, we eventually obtain

$$\int_0^1 \phi_{j1}(\eta)\eta d\eta = \frac{\phi_{j1}(1)}{2} - \frac{\lambda_j}{4}\int_0^1 \phi_{j2}(\eta)\eta d\eta \tag{C6}$$

$$\int_0^1 \phi_{j1}(\eta)\eta^3 d\eta = \frac{\phi_{j1}(1)}{4} - \frac{\lambda_j}{8}\int_0^1 \phi_{j2}(\eta)\eta^3 d\eta \tag{C7}$$

Substituting (C6) and (C7) into (C5) and then into (C4), we have

$$1 = \sum_{j=1}^{\infty} \frac{\phi_{j1}(\eta)\phi_{j1}(1)}{\lambda_j \|\underset{\sim}{\phi}_j\|^2} - \sum_{j=1}^{\infty} \frac{\phi_{j1}(\eta)}{\|\underset{\sim}{\phi}_j\|^2}\int_0^1 h(\eta)\,\phi_{j2}(\eta) d\eta \tag{C8}$$

where

$$h(\eta) = \frac{\eta^3}{2} - \eta \tag{C9}$$

From (34) with

$$\underset{\sim}{f}(\eta) = \begin{bmatrix} 0 \\ \eta h(\eta) \end{bmatrix}$$

and using the first vector components, we obtain

$$\sum_{j=1}^{\infty} \frac{\phi_{j1}(\eta)}{\|\underset{\sim}{\phi}_j\|^2}\int_0^1 h(\eta)\phi_{j2}(\eta) d\eta = 0 \tag{C10}$$

From (C10) and (C8), (C1) is true. This completes the proof.

We shall prove

$$4\sum_{j=1}^{\infty} \frac{\phi_{j1}(\eta)\phi_{j1}(1)}{\lambda_j^2 \|\underset{\sim}{\phi}_j\|^2} = \eta^2 - \frac{\eta^4}{4} + \frac{8}{Pe^2} - \frac{7}{24} \tag{C11}$$

From (25), for $\zeta \to \infty$ and $Z_1 = \infty$, we have

$$\Sigma(\zeta, 1) = 2\int_0^1 [\theta_\infty(\zeta, \eta)\, v(\eta) - \frac{1}{Pe^2}\frac{\partial\theta_\infty}{\partial\zeta}(\zeta, \eta)]\eta d\eta = 2\zeta \tag{C12}$$

where, from (42) and (C1)

$$\theta_\infty(\zeta, \eta) = 4\zeta + 4\sum_{j=1}^{\infty} \frac{\phi_{j1}(1)\phi_{j1}(\eta)}{\lambda_j^2 \|\underset{\sim}{\phi}_j\|^2} \equiv 4\zeta + \Psi(\eta) \tag{C13}$$

Substituting (C13) into (5) and with $v(\eta) = 1 - \eta^2$, we have

$$\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d\Psi}{d\eta}\right) = 4(1 - \eta^2) \tag{C14}$$

from which, solving and by virtue of (7), (8*a*) and (C12), we may finally obtain

$$\Psi(\eta) = \eta^2 - \frac{\eta^4}{4} + \frac{8}{Pe^2} - \frac{7}{24} \tag{C15}$$

which is (C11). This completes the proof.

NOTATION

A^+_j, A^-_j = expansion coefficients, Equations (48) and (49)
$a^{\pm}_j$ = first arguments of Kummer's functions, Equations (69) and (70)
C^+_j, C^-_j = constants, Equations (38) and (39), respectively
c_p = specific heat of the fluid at constant pressure
$D(\ )$ = domain of an operator, Equation (29)
$\underset{\sim}{F}$ = two-component vector in $\mathcal{H}$ identified below Equation (26)
$\underset{\sim}{f}$ = any two-component vector in $\mathcal{H}$, Equation (34)
f = function of η, element of $\mathcal{H}_1$, appendix A
g = function of η, element of $\mathcal{H}_2$, appendix A
h = function of η, Equation (C9)
$\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ = Hilbert spaces, appendix A
k = thermal conductivity of the fluid
$\underset{\sim}{L}$ = matrix differential operator of Equation (26)
$M(\ ,\ ,\ )$ = Kummer's functions, Equations (67) and (68)
Nu = Nusselt number, Equation (58)
Pe = Peclet number, identified below Equation (8)
q_w = uniform wall heat flux, Equation (4)

R = tube radius
r = radial variable
S = axial energy flow function, Equation (9)
T = temperature of the fluid
T_b = bulk temperature of the fluid, Equation (57)
T_0 = temperature of the fluid at $z = -\infty$
T_w = temperature of the fluid at $r=R$, Equation (58)
v_z = fluid velocity, Equation (1)
v = dimensionless fluid velocity identified below Equation (8)
V = some characteristic velocity, identified below Equation (8)
z = axial variable
z_1 = length of heating or mass exchange section, Equation (4)
Z_1 = dimensionless z_1, identified below Equation (8)

Greek Letters

ζ = dimensionless z, identified below Equation (8)
η = dimensionless r, identified below Equation (8)
θ = dimensionless T, identified below Equation (8)
θ_b, θ_w = dimensionless T_b, T_w respectively, Equations (59) and (60)
$\lambda_j, \lambda^+_j, \lambda^-_j$ = eigenvalues, positive and negative eigenvalues, Equations (30) and (33)
μ_j, μ^+_j, μ^-_j = square roots of the absolute values of λ_j, λ^+_j and λ^-_j, respectively, Equation (47)
ρ = density of the fluid
Σ = dimensionless S, Equation (19)
ϕ = vector element of $\mathscr{H}$
$\underset{\sim}{\tilde{\phi}}_j, \underset{\sim}{\phi}^+_j, \underset{\sim}{\phi}^-_j$ = eigenvectors corresponding to λ_j, λ^+_j and λ^-_j, Equation (30)
ψ = vector element of $\mathscr{H}$
$\tilde{\Psi}$ = function of η, equation (C13)
$< , >$ = inner product of Equation (28)
$\| \|$ = norm that corresponds to $< , >$, identified in Equation (35)

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Transport Phenomena in Solids with Bidispersed Pores

Existing analyses for transport phenomena in bidisperse porous media assume that the microparticles act as uniformly distributed point sinks. This article provides an analysis which determines under what conditions the point sink approximation is valid. For random packing, the concentration field inside a pellet is described by its ensemble average, i.e., the average over all possible ways in which the microparticles can be packed into the pellet. For these averaged quantities, we formulate the transport equations for a solid with bidisperse pores which provide the criteria of validity of the point sink approximation.

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SCOPE

Transient response in bidisperse porous media are conventionally analyzed by assuming that individual particles act as point sinks. With the formalism provided here, one can identify the conditions under which such an assumption can be used. The proposed formalism can be used to analyze cases in which the above approximation is not valid.

CONCLUSIONS AND SIGNIFICANCE

Transport equations for a bidispersed system based on an ensemble average have been derived. The concentrations involved in these equations provide a mean performance or response of a large number of pellets in which the microspheres are randomly distributed.

The transient response of bidisperse porous pellets can be classified into two groups.

First, only diffusion takes place, and the steady state or equilibrium profiles of concentrations are respectively linear or uniform in the macropores. The point sink approximation is valid in these cases, if, ω, the ratio between the radius of a microsphere and the major dimension of a pellet, remains below some value, say 1/20. Experimental data obtained previously had been analyzed through the point sink approximation.

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